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# Lacunary Arithmetic convergence

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Abstract. In this article we introduce and study the lacunary arithmetic convergent sequence space $AC_\theta$. Using the idea of strong Cesàro summable sequence and arithmetic convergence we define $AC_{\sigma_1}$ and study the relations between $AC_\theta$ and $AC_{\sigma_1}$. Finally using modulus function we define $AC_\theta(f)$ and study some interesting results.

Key Words: Lacunary sequence; modulus function; arithmetic convergence .

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## 1. Introduction

Throughout, $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ will denote the set of natural, real and complex numbers, respectively and $x = (x_k)$ denotes a sequence whose $k^{th}$ term is $x_k$. Similarly $w, c, \ell_\infty, \ell_1$ denotes the space of *all, convergent, bounded, absolutely summable* sequences of complex terms, respectively.

We use the symbol $< m, n >$ to denote the greatest common divisor of two integers $m$ and $n$.
W.H.Ruckle [11], introduced the notions arithmetic convergence as a sequence $x = (x_m)$ is called *arithmetically convergent* if for each $\varepsilon > 0$ there is an integer $n$ such that for every integer $m$ we have $|x_m - x_{<m,n>}| < \varepsilon$. We denote the sequence space of all arithmetic convergent sequence by $AC$. The studies on arithmetic convergence and related results can be found in [11, 13, 14, 15].

The notion of a modulus function was introduced in 1953 by Nakano [9]. We recall [7, 10] that a modulus $f$ is a function $f : [0, \infty) \to [0, \infty)$ such that

(i) $f(x) = 0$ if and only if $x = 0$,
(ii) $f(x + y) \le f(x) + f(y)$ for all $x \ge 0, y \ge 0$,
(iii) $f$ is increasing,
(iv) $f$ is continuous from the right at 0.

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Because of (ii), $|f(x)-f(y)| \leq f(|x-y|)$ so that in view of (iv), $f$ is continuous everywhere on $[0,\infty)$. A modulus may be unbounded (for example, $f(x)=x^p, 0<p\leq 1$) or bounded (for example, $f(x)=\frac{x}{1+x}$).
It is easy to see that $f_1+f_2$ is a modulus function when $f_1$ and $f_2$ are modulus functions, and that the function $f_i$($i$ is a positive integer), the composition of a modulus function $f$ with itself $i$ times, is also a modulus function.
Ruckle [12] used the idea of a modulus function $f$ to construct a class of FK spaces

$$X(f)=\left\{x=(x_k):\sum_{k=1}^{\infty} f(|x_k|)<\infty\right\}.$$

The space $X(f)$ is closely related to the space $l_1$ which is an $X(f)$ space with $f(x)=x$ for all real $x\geq 0$. Thus Ruckle [12] proved that, for any modulus $f$, $X(f)\subset l_1$. The space $X(f)$ is a Banach space with respect to the norm $\|x\|=\sum_{k=1}^{\infty} f(|x_k|)<\infty$.
Spaces of the type $X(f)$ are a special case of the spaces structured by Gramsch in [6]. By a lacunary sequence we mean an increasing integer sequence $\theta=(k_r)$ such that $h_r=k_r-k_{r-1}\to\infty$ as $r\to\infty$. In this paper the intervals determined by $\theta$ will be denoted by $I_r=(k_{r-1},k_r]$ and also the ratio $\frac{k_r}{k_{r-1}}, r\geq 1, k_0\neq 0$ will be denoted by $q_r$. The space of lacunary convergence sequence $N_\theta$ was defined by Freedman [3] as follows:

$$N_\theta=\left\{x=(x_i)\in w:\lim_{r\to\infty}\frac{1}{h_r}\sum_{i\in I_r}|x_i-l|=0 \text{ for some } l\right\}.$$

The space $N_\theta$ is a $BK$-space with the norm

$$\|x\|_{N_\theta}=\sup_r \frac{1}{h_r}\sum_{i\in I_r}|x_i|.$$

The notion of lacunary convergence has been investigated by Çolak [2], Fridy and Orhan [4, 5], Tripathy and Et [16] and many others in the recent past.

The main purpose of this paper is to introduce and study the concept of lacunary arithmetic convergence.

## 2. Lacunary Arithmetic Convergence

In this section we introduce the lacunary arithmetic convergent sequence space $AC_\theta$ as follows:

$$AC_\theta=\left\{(x_m):\lim_{r\to\infty}\frac{1}{h_r}\sum_{m\in I_r}|x_m-x_{<m,n>}|=0 \text{ for some integer } n\right\}.$$

**Theorem 2.1.** *The sequence space $AC_\theta$ is linear.*

*Proof.* Let $(x_m)$ and $(y_m)$ be two sequences in $AC_\theta$. Then for an integer $n$

$$\lim_{r\to\infty}\frac{1}{h_r}\sum_{m\in I_r}|x_m-x_{<m,n>}|=0 \text{ and } \lim_{r\to\infty}\frac{1}{h_r}\sum_{m\in I_r}|y_m-y_{<m,n>}|=0.$$

Let $\alpha$ and $\beta$ be two scalars, then there exist integers $T_\alpha$ and $M_\beta$ such that $|\alpha| \leq T_\alpha$ and $|\beta| \leq M_\beta$. Thus

$$\frac{1}{h_r}\sum_{m\in I_r}|\alpha x_m + \beta y_m - (\alpha x_{<m,n>} + \beta y_{<m,n>})|$$
$$\leq T_\alpha \frac{1}{h_r}\sum_{m\in I_r}|x_m - x_{<m,n>}| + M_\beta \frac{1}{h_r}\sum_{m\in I_r}|y_m - y_{<m,n>}|$$

which implies that $\alpha x_m + \beta y_m \to \alpha x_{<m,n>} + \beta y_{<m,n>}$.
Hence $AC_\theta$ is linear. □

**Theorem 2.2.** *If $(x_m)$ is a sequence in $AC$ then $(x_m)$ is a sequence in $AC_\theta$.*

*Proof.* Let $(x_m)$ be a sequence in $AC$. Then for $\varepsilon > 0$ there is an integer $n$ such that

$$|x_m - x_{<m,n>}| < \varepsilon.$$

Now, for an integer $n$, we have

$$\begin{aligned}\frac{1}{h_r}\sum_{m\in I_r}|x_m - x_{<m,n>}| &= \frac{1}{h_r}\left[\sum_{m=1}^{k_r}|x_m - x_{<m,n>}| - \sum_{m=1}^{k_{r-1}}|x_m - x_{<m,n>}|\right]\\ &< \frac{1}{h_r}(h_r\varepsilon)\\ &= \varepsilon.\end{aligned}$$

Thus $(x_m) \in AC_\theta$. □

**Definition 2.3.** [3] Let $\theta = (k_r)$ be a lacunary sequence. A lacunary refinement of $\theta$ is a lacunary sequence $\theta' = (k'_r)$ satisfying $(k_r) \subseteq (k'_r)$.

**Theorem 2.4.** *If $\theta'$ is a lacunary refinement of a lacunary sequence $\theta$ and $(x_m) \in AC_{\theta'}$ then $AC_\theta$.*

*Proof.* Suppose for each $I_r$ of $\theta$ contains the point $(k'_{r,t})_{t=1}^{\eta(r)}$ of $\theta'$ such that

$$k_{r-1} < k'_{r,1} < k'_{r,2} < \ldots < k'_{\eta,\eta(r)} = k_r,$$

where $I'_{r,t} = \left(I'_{r,t-1}, I'_{r,t}\right]$.
Since $(k_r) \subseteq (k'_r)$, so $r, \eta(r) \geq 1$.
Let $(I^*)_{j=1}^{\infty}$ be the sequence of interval $(I^*_{r,t})$ ordered by increasing right end points. Since $(x_m) \in AC_{\theta'}$, then for each $\varepsilon > 0$,

$$\frac{1}{h^*_j}\sum_{I^*_j \subset I_r}|x_m - x_{<m,n>}| < \varepsilon.$$

Also, since $h_r = k_r - k_{r-1}$, so $h'_{r,t} = k'_{r,t} - k'_{r,t-1}$.
For each $\varepsilon > 0$,

$$\frac{1}{h_r}\sum_{m\in I_r}|x_m - x_{<m,n>}| \leq \frac{1}{h^*_j}\sum_{I^*_j \subset I_r}|x_m - x_{<m,n>}| < \varepsilon.$$

This implies $(x_m) \in AC_\theta$. □

Based on the idea of strongly Cesàro summable sequences and arithmetic convergent sequences, we introduce a new sequence space $AC_{\sigma_1}$ defined as follows:

$$AC_{\sigma_1} = \left\{ (x_m) : \text{there exists an integer } n \text{ such that } \frac{1}{t}\sum_{m=1}^{t} |x_m - x_{<m,n>}| \to 0 \text{ as } t \to \infty \right\}.$$

Functional analytic studies of the space $|\sigma_1|$ of strongly Cesàro summable sequences, and other closely related spaces can be found in [1, 8].
In this section we shall mostly focus on the connection between the spaces $AC_\theta$ and $AC_{\sigma_1}$.

**Theorem 2.5.** *The sequence space $AC_{\sigma_1}$ is a linear space.*

*Proof.* The proof is a routine exercise and hence ommited. □

**Theorem 2.6.** *Let $\theta = (k_r)$ be a lacunary sequence. If $\liminf q_r > 1$ then $AC_{\sigma_1} \subseteq AC_\theta$.*

*Proof.* Let $(x_m) \in AC_{\sigma_1}$ and $\liminf q_r > 1$. Then there exists $\delta > 0$ such that $q_r = \frac{k_r}{k_{r-1}} \geq 1+\delta$ for sufficiently large $r$. We can also choose a sufficiently large $r$ so that $\frac{k_r}{h_r} \leq \frac{1+\delta}{\delta}$. Then

$$\begin{aligned}
&\frac{1}{k_r}\sum_{m=1}^{k_r} |x_m - x_{<m,n>}| \\
&\geq \frac{1}{k_r}\sum_{m\in I_r} |x_m - x_{<m,n>}| \\
&= \frac{h_r}{k_r}\left( h_r^{-1}\sum_{m\in I_r} |x_m - x_{<m,n>}| \right) \\
&\geq \frac{\delta}{1+\delta}\left( h_r^{-1}\sum_{m\in I_r} |x_m - x_{<m,n>}| \right)
\end{aligned}$$

which proves that $(x_m) \in AC_\theta$. □

**Theorem 2.7.** *For $\limsup q_r < \infty$, we have $AC_\theta \subseteq AC_{\sigma_1}$.*

*Proof.* Let $\limsup q_r < \infty$ then there exists $K > 0$ such that $q_r < K$ for every $r$. Now for $\varepsilon > 0$ and $(x_m) \in AC_\theta$ there exists $R$ such that for every $r \geq R$,

$$\tau_r = \frac{1}{h_r}\sum_{m\in I_r} |x_m - x_{<m,n>}| < \varepsilon.$$

We can also find $T > 0$ such that $\tau_r \leq T$ $\forall r$. Let $t$ be any integer with $k_r \geq t \geq k_{r-1}$. Then

$$\begin{aligned}
&\frac{1}{t}\sum_{m=1}^{t}|x_m - x_{<m,n>}|\\
&\leq \frac{1}{k_{r-1}}\sum_{m=1}^{k_r}|x_m - x_{<m,n>}|\\
&= \frac{1}{k_{r-1}}\sum_{i=1}^{R}\sum_{m\in I_i}|x_m - x_{<m,n>}| + \frac{1}{k_{r-1}}\sum_{i=R+1}^{k_r}\sum_{m\in I_i}|x_m - x_{<m,n>}|\\
&\leq \frac{1}{k_{r-1}}\sum_{i=1}^{R}\sum_{m\in I_i}|x_m - x_{<m,n>}| + \frac{1}{k_{r-1}}\left(\varepsilon(k_r - k_R)\right)\\
&\leq \frac{1}{k_{r-1}}\sum_{i=1}^{R}h_i\tau_i + \frac{1}{k_{r-1}}\left(\varepsilon(k_r - k_R)\right)\\
&\leq \frac{1}{k_{r-1}}\left(\sup_{i\leq R}\tau_i k_R\right) + \varepsilon K\\
&< \frac{k_R}{k_{r-1}}T + \varepsilon K
\end{aligned}$$

from which we deduce that $(x_m) \in AC_{\sigma_1}$. □

The following corollary follows from Threoms 2.6 and 2.7.

**Corollary 2.8.** *If* $1 < \liminf q_r \leq \limsup q_r < \infty$, *then* $AC_\theta = AC_{\sigma_1}$.

Next we introduce the lacunary arithmetic convergent sequence space $AC_\theta(f)$ defined by modulus function $f$.
Let $f$ be a modulus function. We define

$$AC_\theta(f) = \left\{(x_m) : \text{there exists an integer } n \text{ such that } \frac{1}{h_r}\sum_{m\in I_r} f(|x_m - x_{<m,n>}|) \to 0 \text{ as } r \to \infty\right\}.$$

Note that if we put $f(x) = x$, then $AC_\theta(f) = AC_\theta$.

**Theorem 2.9.** *The sequence space* $AC_\theta(f)$ *is a linear space.*

*Proof.* Let $(x_m)$ and $(y_m)$ be two sequences in $AC_\theta(f)$. Then for an integer $n$ and $\varepsilon > 0$,

$$\frac{1}{h_r}\sum_{m\in I_r} f(|x_m - x_{<m,n>}|) \to 0 \text{ and } \frac{1}{h_r}\sum_{m\in I_r} f(|y_m - y_{<m,n>}|) \to 0 \text{ as } r \to \infty.$$

Let $\alpha$ and $\beta$ be two scalars, then there exist integers $T_\alpha$ and $M_\beta$ such that $|\alpha| \leq T_\alpha$ and $|\beta| \leq M_\beta$. Thus

$$\begin{aligned}
&\frac{1}{h_r}\sum_{m\in I_r} f(|\alpha x_m + \beta y_m - (\alpha x_{<m,n>} + \beta y_{<m,n>})|)\\
&\leq T_\alpha \frac{1}{h_r}\sum_{m\in I_r} f(|x_m - x_{<m,n>}|) + M_\beta \frac{1}{h_r}\sum_{m\in I_r} f(|y_m - y_{<m,n>}|)\\
&\to 0 \text{ as } r\to\infty
\end{aligned}$$

which implies that $\alpha x_m + \beta y_m \to \alpha x_{<m,n>} + \beta y_{<m,n>}$ in $AC_\theta(f)$.
Hence $AC_\theta(f)$ is linear. □

**Proposition 2.10.** [10] *Let $f$ be a modulus and let $0 < \delta < 1$. Then for each $x \geq \delta$, we have $f(x) \leq 2f(1)\delta^{-1}x$.*

**Theorem 2.11.** *Let $f$ be any modulus such that $\lim_{t\to\infty}\frac{f(t)}{t} = \beta > 0$ then $AC_\theta(f) = AC_\theta$.*

*Proof.* Let $(x_m) \in AC_\theta$, then for an integer $n$,

$$\tau_r = \frac{1}{h_r}\sum_{m\in I_r} |x_m - x_{<m,n>}| \to 0 \text{ as } r\to\infty.$$

Let $\varepsilon > 0$ be given. We choose $0 < \delta < 1$ such that $f(u) < \varepsilon$ for every $u$ with $0 \leq u \leq \delta$. We can write

$$\begin{aligned}
&\frac{1}{h_r}\sum_{m\in I_r} f(|x_m - x_{<m,n>}|)\\
&= \frac{1}{h_r}\sum_{\substack{m\in I_r;\\ |x_m - x_{<m,n>}|\leq\delta}} f(|x_m - x_{<m,n>}|) + \frac{1}{h_r}\sum_{\substack{m\in I_r;\\ |x_m - x_{<m,n>}|>\delta}} f(|x_m - x_{<m,n>}|)\\
&\leq \frac{1}{h_r}(h_r\varepsilon) + 2f(1)\delta^{-1}\tau_r, \text{ using (Proposition 2.10)}\\
&\to 0 \text{ as } r\to\infty.
\end{aligned}$$

Therefore $(x_m) \in AC_\theta(f)$.
Till this part of the proof we do need $\beta > 0$. Now let $\beta > 0$ and let $(x_m) \in AC_\theta(f)$. Since $\beta > 0$, we have $f(t) \geq \beta t$ $\forall t \geq 0$. Hence it follows that $(x_m) \in AC_\theta$. □

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